DYNAMICS OF TUPLES OF MATRICES

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Abstract. In this article we answer a question raised by N. Feldman in [4] concerning the dynamics of tuples of operators on $\mathbb{R}^n$. In particular, we prove that for every positive integer $n \geq 2$ there exist n tuples $(A_1, A_2, \ldots, A_n)$ of $n \times n$ matrices over $\mathbb{R}$ such that $(A_1, A_2, \ldots, A_n)$ is hypercyclic. We also establish related results for tuples of 2×2 matrices over $\mathbb{R}$ or $\mathbb{C}$ being in Jordan form.

1. Introduction

Following the recent work of Feldman in [4], an n-tuple of operators is a finite sequence of length n of commuting continuous linear operators $T_1, T_2, \cdots, T_n$ acting on a locally convex space X. The tuple $(T_1, T_2, \ldots, T_n)$ is hypercyclic if there exists a vector $x \in X$ such that the set

$$\{T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} x : k_1, k_2, \ldots k_n \geq 0\}$$

is dense in X. Such a vector x is called hypercyclic for $(T_1, T_2, \ldots, T_n)$ and the set of hypercyclic vectors for $(T_1, T_2, \ldots, T_n)$ will be denoted by $HC((T_1, T_2, \ldots, T_n))$. The above definition generalizes the notion of hypercyclicity to tuples of operators. For an account of results, comments and an extensive bibliography on hypercyclicity we refer to [1], [5], [6] and [7]. For results concerning the dynamics of tuples of operators see [2], [3], [4] and [8].

In [4] Feldman showed, among other things, that in $\mathbb{C}^n$ there exist diagonalizable $n+1$-tuples of matrices having dense orbits. In addition he proved that there is no n-tuple of diagonalizable matrices on $\mathbb{R}^n$ or $\mathbb{C}^n$ that has a somewhere dense orbit. Therefore the following question arose naturally.

Question (Feldman [4]**).** *Are there non-diagonalizable n-tuples on $\mathbb{R}^k$ that have somewhere dense orbits*?

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We give a positive answer to this question in a very strong form, as the next theorem shows.

Theorem 1.1. *For every positive integer $n \geq 2$ there exist n-tuples $(A_1, A_2, \dots, A_n)$ of $n \times n$ non-diagonalizable matrices over $\mathbb{R}$ such that $(A_1, A_2, \dots, A_n)$ is hypercyclic.*

Restricting ourselves to tuples of 2×2 matrices in Jordan form either on $\mathbb{R}^2$ or $\mathbb{C}^2$, we prove the following.

Theorem 1.2. *There exist 2×2 matrices $A_j, j = 1, 2, 3, 4$ in Jordan form over $\mathbb{R}$ such that (A_1, A_2, A_3, A_4) is hypercyclic. In particular*

$$HC((A_1, A_2, A_3, A_4)) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\}.$$

Theorem 1.3. *There exist 2×2 matrices $A_j, j = 1, 2, \dots, 8$ in Jordan form over $\mathbb{C}$ such that $(A_1, A_2, \dots, A_8)$ is hypercyclic.*

2. Products of 2×2 matrices

Lemma 2.1. *Let m be a positive integer and for each $j = 1, 2, \dots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$, i.e. $A_j = \begin{pmatrix} a_j & 1 \\ 0 & a_j \end{pmatrix}$ for $a_1, a_2, \dots, a_m \in \mathbb{F}$. Then $(A_1, A_2, \dots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic if and only if the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \dots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \dots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \dots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$ (respectively $\mathbb{R}^2$).

Proof. We prove the above in the case $\mathbb{F} = \mathbb{C}$, since the other case is similar. Observe that

$${A_j}^l = \begin{pmatrix} {a_j}^l & l{a_j}^{l-1} \\ 0 & {a_j}^l \end{pmatrix}$$

for $l \in \mathbb{N}$. As a result we have

$${A_1}^{k_1} {A_2}^{k_2} \dots {A_m}^{k_m} = \begin{pmatrix} \prod_{j=1}^{m} {a_j}^{k_j} & \prod_{j=1}^{m} {a_j}^{k_j} \sum_{s=1}^{m} \frac{k_s}{a_s} \\ 0 & \prod_{j=1}^{m} {a_j}^{k_j} \end{pmatrix}.$$

Assume that $(A_1, A_2, \dots, A_m)$ is hypercyclic and let $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$ be a hypercyclic vector for $(A_1, A_2, \dots, A_m)$. Then the sequence

$$\left\{ {A_1}^{k_1} {A_2}^{k_2} \dots {A_m}^{k_m} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} : k_1, k_2, \dots, k_m \in \mathbb{N} \right\}$$

$$= \left\{ \begin{pmatrix} z_1 \prod_{j=1}^{m} {a_j}^{k_j} + z_2 \prod_{j=1}^{m} {a_j}^{k_j} \sum_{s=1}^{m} \frac{k_s}{a_s} \\ z_2 \prod_{j=1}^{m} {a_j}^{k_j} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. This implies that $z_2 \neq 0$. Dividing the element on the first row by that on the second, it can easily be shown that the sequence

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_m}{a_m} \\ {a_1}^{k_1} {a_2}^{k_2} \ldots {a_m}^{k_m} \end{pmatrix} : k_1, k_2, \ldots, k_m \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. The converse can easily be shown. □

Remark 2.2. Let m be a positive integer and for each $j = 1, 2, \ldots, m$ let A_j be a 2×2 matrix in Jordan form over a field $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$. By the proof of Lemma 2.1 it is immediate that whenever $(A_1, A_2, \ldots, A_m)$ over $\mathbb{C}$ (respectively $\mathbb{R}$) is hypercyclic one can completely describe the set of hypercyclic vectors as

$$\left\{ \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2 : z_2 \neq 0 \right\} \quad \left(\text{respectively } \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_2 \neq 0 \right\}. \right)$$

2.1. **The real case.** We shall need the following well known result, see for example [4].

Theorem 2.3. *If $a, b > 1$ and $\frac{\ln a}{\ln b}$ is irrational then the sequence $\{\frac{a^n}{b^m} : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$.*

Lemma 2.4. *Let $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $b > 1$ and $\frac{\ln |a|}{\ln b}$ is irrational. Then the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$.*

Proof. Since $\frac{\ln |a|}{\ln b}$ is irrational it follows that $\ln b / \ln \frac{1}{a^2}$ is irrational as well. Applying Theorem 2.3 we conclude that the sequence $\{a^{2n} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^+$. On the other hand the fact that a is negative implies that the sequence $\{a^{2n+1} b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}^-$. This completes the proof of the lemma. □

In what follows, for any $x \in \mathbb{R}$ we will be denoting by $[x]$ the 'integral part of x', that is, the largest integer which does not exceed x, and by $\{x\}$ the 'fractional part of x', that is, $\{x\} = x - [x]$.

Proposition 2.5. *There exist $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \frac{k_3}{a_3} + \frac{k_4}{a_4} \\ {a_1}^{k_1} {a_2}^{k_2} {a_3}^{k_3} {a_4}^{k_4} \end{pmatrix} : k_1, k_2, k_3, k_4 \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$.

Proof. By the lemma above fix $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $a + \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{R}$. Let $x_1, x_2 \in \mathbb{R}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - x_2| < \epsilon$. Note that $a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s$ for every $k, s \in \mathbb{N}$. Note also that $a + \frac{1}{a} < 0$. In the case $x_1 \geq 0$ (resp. $x_1 < 0$) there exists $k \in \mathbb{N}$ such that $\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right)$ is less than -1 (resp. $x_1 - 1$) and

$$\left| \left\{ \frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) \right\} - \{x_1\} \right| < \epsilon.$$

As there exists $s \in \mathbb{N}$ such that

$$\left| \frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) + s - x_1 \right| < \epsilon$$

we are done. Hence, setting $a_1 = a, a_2 = b, a_3 = \frac{1}{a}, a_4 = 1$ the result is proved. $\square$

Proof of Theorem 1.2. This is an immediate consequence of Lemma 2.1, Proposition 2.5 and Remark 2.2.

Example 2.6. One may construct many concrete examples of four 2×2 matrices, in Jordan form over $\mathbb{R}$, being hypercyclic. For example, fix $a, b \in \mathbb{R}$ such that $-1 < a < 0$, $b > 1$ and both $a + \frac{1}{a}$, $\frac{\ln|a|}{\ln b}$ are irrational. From the above we conclude that

$$\left(\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \begin{pmatrix} \frac{1}{a} & 1 \\ 0 & \frac{1}{a} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

is hypercyclic.

Proposition 2.7. *(i) Every pair (A_1, A_2) of 2×2 matrices over $\mathbb{R}$ with A_j, $j = 1, 2$ being either diagonal or in Jordan form is not hypercyclic.*

(ii) There exist pairs (A_1, A_2) of 2×2 matrices over $\mathbb{R}$ such that A_1 is diagonal, A_2 is antisymmetric (rotation matrix) and (A_1, A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1, A_2), i.e.

$$HC((A_1, A_2)) = \mathbb{R}^2 \setminus \{(0, 0)\}.$$

(iii) There exist pairs (A_1, A_2) of 2×2 matrices over $\mathbb{R}$ such that both A_1 and A_2 are antisymmetric and (A_1, A_2) is hypercyclic. In particular every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1, A_2), i.e.

$$HC((A_1, A_2)) = \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Proof. Let us prove assertion (i). The case of A_1, A_2 both diagonal is covered by Feldman, see [4]. Assume that A_1 is diagonal and A_2 is in Jordan form, i.e. $A_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $\quad A_2 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}$ for $a, b \in$

$\mathbb{R}$. Suppose that (A_1, A_2) is hypercyclic and let $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$ be a hypercyclic vector for (A_1, A_2). Then the sequence

$$\left\{ {A_1}^n {A_2}^m \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : n, m \in \mathbb{N} \right\},$$

i.e. the sequence $\left\{ \begin{pmatrix} a^n b^m x_1 + m a^n b^m x_2 \\ a^n b^m x_2 \end{pmatrix} : n, m \in \mathbb{N} \right\}$ is dense in $\mathbb{R}^2$. Observe that x_2 cannot be zero. Then there exist $y_1, y_2 \in \mathbb{R}$ and sequences of positive integers $\{n_k\}$, $\{m_k\}$ such that $m_k \to +\infty$ and

$$a^{n_k} b^{m_k} x_1 + m_k a^{n_k} b^{m_k} x_2 \to y_1,$$
$$a^{n_k} b^{m_k} x_2 \to y_2$$

as $k \to +\infty$. It clearly follows that $|m_k a^{n_k} b^{m_k} x_2| \to +\infty$ which is a contradiction. Assume now that both A_1, A_2 are in Jordan form, i.e.

$$A_1 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix},$$

for $a, b \in \mathbb{R}$ and (A_1, A_2) is hypercyclic. Lemma 2.1 implies that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ a^n b^m \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Observe that both $|a|, |b|$ are not equal to 1. By taking absolute value in the second coordinate and then applying the logarithmic function, it follows that the sequence

$$\left\{ \begin{pmatrix} \frac{n}{a} + \frac{m}{b} \\ n \ln|a| + m \ln|b| \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence the sequence

$$\left\{ \begin{pmatrix} n\frac{\ln|a|}{a} + m\frac{\ln|a|}{b} \\ n\frac{\ln|a|}{a} + m\frac{\ln|b|}{a} \end{pmatrix} : n, m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Subtracting the second coordinate from the first one we conclude that the sequence

$$\left\{ m \left(\frac{\ln|a|}{b} - \frac{\ln|b|}{a} \right) : m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}$ which is absurd. We proceed with the proof of assertion (ii). There exist $a \in \mathbb{R} \setminus \mathbb{Q}$ and $b \in \mathbb{C}$ such that the sequence $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$, see [4]. Write $b = |b| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b|\cos\theta & -|b|\sin\theta \\ |b|\sin\theta & |b|\cos\theta \end{pmatrix}.$$

Then we have

$$A_1{}^n A_2{}^m = \begin{pmatrix} a^n|b|^m\cos(m\theta) & -a^n|b|^m\sin(m\theta) \\ a^n|b|^m\sin(m\theta) & a^n|b|^m\cos(m\theta) \end{pmatrix}.$$

Applying in the above relation the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and taking into account that the sequence $\{a^n b^m : n,m \in \mathbb{N}\}$ is dense in $\mathbb{C}$ we conclude that the sequence

$$\left\{ A_1{}^n A_2{}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n,m \in \mathbb{N} \right\} = \left\{ \begin{pmatrix} a^n|b|^m\cos(m\theta) \\ a^n|b|^m\sin(m\theta) \end{pmatrix} : n,m \in \mathbb{N} \right\}$$

is dense in $\mathbb{R}^2$. Hence (A_1, A_2) is hypercyclic. It is now easy to show that every non-zero vector in $\mathbb{R}^2$ is hypercyclic for (A_1, A_2).

In order to prove the last assertion we follow a similar line of reasoning as above. That is, by Corollary 3.2 in [4] there exist $a, b \in \mathbb{C}\setminus\mathbb{R}$ such that the sequence $\{a^n b^m : n,m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. Write $a = |a|e^{i\phi}$, $b = |b|e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a|\cos\phi & -|a|\sin\phi \\ |a|\sin\phi & |a|\cos\phi \end{pmatrix}, \quad A_2 = \begin{pmatrix} |b|\cos\theta & -|b|\sin\theta \\ |b|\sin\theta & |b|\cos\theta \end{pmatrix}.$$

A direct computation gives that $\left\{ A_1{}^n A_2{}^m \begin{pmatrix} 1 \\ 0 \end{pmatrix} : n,m \in \mathbb{N} \right\}$ equals to $\left\{ \begin{pmatrix} |a|^n|b|^m\cos(n\phi - m\theta) \\ |a|^n|b|^m\sin(n\phi + m\theta) \end{pmatrix} : n,m \in \mathbb{N} \right\}$ and by the choice of a, b we conclude that the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is hypercyclic for (A_1, A_2). This completes the proof of the proposition. □

Question 2.8. *What is the minimum number of 2×2 matrices over $\mathbb{R}$ in Jordan form so that their tuple form a hypercyclic operator?*

2.2. **The complex case.** In what follows we will be writing $\Re(z)$ and $\Im(z)$ for the real and imaginary parts of a complex number z respectively.

Proposition 2.9. *There exist $a_j \in \mathbb{C}$, $j = 1, 2, \ldots, 8$ such that the sequence*

$$\left\{ \begin{pmatrix} \frac{k_1}{a_1} + \frac{k_2}{a_2} + \ldots + \frac{k_8}{a_8} \\ a_1{}^{k_1} a_2{}^{k_2} \ldots a_8{}^{k_8} \end{pmatrix} : k_1, k_2, \ldots, k_8 \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

Proof. The proof is in the same spirit as the proof of Proposition 2.5. Fix $a, b \in \mathbb{C}$ such that $-1 < a < 0$, $a + \frac{1}{a}, a - \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m :$

$n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$ (see Corollary 3.2). Let $z_1, z_2 \in \mathbb{C}$ and $\epsilon > 0$ be given. Then there exist $n, m \in \mathbb{N}$ such that $|a^n b^m - z_2| < \epsilon$. Note that

$$a^n b^m = a^{n+k} b^m \frac{1}{a^k} 1^s (ia)^\xi \left(\frac{1}{ia}\right)^\xi (4i)^\rho \left(-\frac{1}{4}\right)^\rho$$

for every $k, s, \xi \in \mathbb{N}$ and $\rho \in 4\mathbb{N}$. Note that $a + \frac{1}{a} < 0$ and $a - \frac{1}{a} > 0$. In the case $\Re(z_1) \geq 0$ (resp. $\Re(z_1) < 0$) there exists $k \in \mathbb{N}$ such that $\Re\left(\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right)\right)$ is less than -1 (resp. $\Re(z_1) - 1$) and

$$\left|\left\{\Re\left(\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right)\right)\right\} - \{\Re(z_1)\}\right| < \epsilon.$$

Also, in the case $\Im(z_1) \geq 0$ (resp. $\Im(z_1) < 0$) there exists $\xi \in \mathbb{N}$ such that $\Im\left(\frac{n}{a} + \frac{m}{b} + i\xi\left(a - \frac{1}{a}\right)\right)$ is greater than 1 (resp. $\Im(z_1) + 1$) and

$$\left|\left\{\Im\left(\frac{n}{a} + \frac{m}{b} + i\xi\left(a - \frac{1}{a}\right)\right)\right\} - \{\Im(z_1)\}\right| < \epsilon.$$

As there exists $\rho \in 4\mathbb{N}$ such that

$$\left|\Im\left(\frac{n}{a} + \frac{m}{b} + i\xi\left(a - \frac{1}{a}\right) - \rho\left(\frac{i}{4}\right)\right) - \Im(z_1)\right| < \epsilon$$

we are done with the imaginary part. Also, there exists $s \in \mathbb{N}$ such that

$$\left|\Re\left(\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) - 4\rho + s\right) - \Re(z_1)\right| < \epsilon.$$

But this means that the real and imaginary parts of the complex number

$$\frac{n}{a} + \frac{m}{b} + k\left(a + \frac{1}{a}\right) + s + i\xi\left(a - \frac{1}{a}\right) - \rho\frac{i}{4} - 4\rho$$

are within ϵ of the real and imaginary parts of z_1. Hence, setting $a_1 = a, a_2 = b, a_3 = \frac{1}{a}, a_4 = 1, a_5 = ia, a_6 = \frac{1}{ia}, a_7 = 4i, a_8 = -\frac{1}{4}$ the result is proved. □

Proof of Theorem 1.3. By Proposition 2.9, Lemma 2.1 and Remark 2.2 the assertion follows.

Example 2.10. Fix $a, b \in \mathbb{C}$ such that $-1 < a < 0$, $a + \frac{1}{a}, a - \frac{1}{a} \in \mathbb{R} \setminus \mathbb{Q}$ and $\{a^n b^m : n, m \in \mathbb{N}\}$ is dense in $\mathbb{C}$. From the above it is evident that the following 8-tuple of 2×2 matrices in Jordan form over $\mathbb{C}$

$$\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}, \begin{pmatrix} b & 1 \\ 0 & b \end{pmatrix}, \begin{pmatrix} \frac{1}{a} & 1 \\ 0 & \frac{1}{a} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} ia & 1 \\ 0 & ia \end{pmatrix}, \begin{pmatrix} \frac{1}{ia} & 1 \\ 0 & \frac{1}{ia} \end{pmatrix}, \begin{pmatrix} 4i & 1 \\ 0 & 4i \end{pmatrix}, \begin{pmatrix} -\frac{1}{4} & 1 \\ 0 & -\frac{1}{4} \end{pmatrix}$$

is hypercyclic.

Question 2.11. *What is the minimum number of* 2×2 *matrices over* $\mathbb{C}$ *in Jordan form so that their tuple form a hypercyclic operator?*

3. Products of 3×3 matrices

Proposition 3.1. (Feldman) *If* $b_1, b_2 \in \mathbb{D} \setminus \{0\}$ *then there exists a dense set* $\Delta \subset \mathbb{C} \setminus \mathbb{D}$ *such that for every* $a_1, a_2 \in \Delta$ *the sequence*

$$\left\{ \begin{pmatrix} {a_1}^n {b_1}^m \\ {a_2}^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$.

Corollary 3.2. *There exist* $a \in \mathbb{C}$ *and* $b, c, d \in \mathbb{R}$ *such that the sequence*

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times \mathbb{R}$.

Proof. Fix two real numbers b_1, b_2 with $b_1, b_2 \in (0, 1)$. By Proposition 3.1 there exist $a_1, a_2 \in \mathbb{C} \setminus \mathbb{D}$ such that the sequence

$$\left\{ \begin{pmatrix} {a_1}^n {b_1}^m \\ {a_2}^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C}^2$. Define $a = a_1$, $b = b_1$, $c = |a_2|$ and $d = -\sqrt{b_2}$. Observe that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n {b_2}^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times [0, +\infty)$. Take $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

Case I: $x \geq 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}$, $\{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to x.$$

Since ${b_2}^{l_k} = d^{2l_k}$ we get $c^{n_k} d^{2l_k} \to x$.

Case II: $x < 0$.

Then there exist sequences of positive integers $\{n_k\}, \{m_k\}$, $\{l_k\}$ such that

$$a^{n_k} b^{m_k} \to z \text{ and } c^{n_k} {b_2}^{l_k} \to \frac{x}{d}.$$

The last implies that $c^{n_k} d^{2l_k+1} \to x$. This completes the proof of the corollary. ☐

Proposition 3.3. *There exist* 3 *tuples* (A_1, A_2, A_3) *of* 3×3 *matrices over* $\mathbb{R}$ *such that* (A_1, A_2, A_3) *is hypercyclic.*

Proof. By Corollary 3.2 there exist $a \in \mathbb{C}$ and $b, c, d \in \mathbb{R}$ such that the sequence

$$\left\{ \begin{pmatrix} a^n b^m \\ c^n d^l \end{pmatrix} : n, m, l \in \mathbb{N} \right\}$$

is dense in $\mathbb{C} \times \mathbb{R}$. Write $a = |a| e^{i\theta}$ and set

$$A_1 = \begin{pmatrix} |a|\cos\theta & -|a|\sin\theta & 0 \\ |a|\sin\theta & |a|\cos\theta & 0 \\ 0 & 0 & c \end{pmatrix}, \; A_2 = \begin{pmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and}$$

$A_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{pmatrix}$. Then we have

$$A_1{}^n A_2{}^m A_3{}^l = \begin{pmatrix} |a|^n b^m \cos(n\theta) & -|a|^n b^m \sin(n\theta) & 0 \\ |a|^n b^m \sin(n\theta) & |a|^n b^m \cos(n\theta) & 0 \\ 0 & 0 & c^n d^l \end{pmatrix},$$

which in turn gives

$$A_1{}^n A_2{}^m A_3{}^l \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} |a|^n b^m \cos(n\theta) \\ |a|^n b^m \sin(n\theta) \\ c^n d^l \end{pmatrix}.$$

The last and the choice of a, b, c, d imply that (A_1, A_2, A_3) is hypercyclic with $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ being a hypercyclic vector for (A_1, A_2, A_3). $\square$

4. Proof of Theorem 1.1

By Proposition 2.7, there exist 2×2 matrices B_1 and B_2 such that (B_1, B_2) is hypercyclic.

Case I: $n = 2k$ for some positive integer k. For $k = 1$ the result follows by Proposition 2.7. Assume that $k > 1$. Each A_j will be constructed by blocks of 2×2 matrices. Let I_2 be the 2×2 identity matrix. We will be using the notation $diag(D_1, D_2, \cdots, D_n)$ to denote the diagonal matrix with diagonal entries the block matrices $D_1, D_2, \cdots, D_n$. Define $A_1 = diag(B_1, I_2, \cdots, I_2)$, $A_2 = diag(B_2, I_2, \cdots, I_2)$, $A_3 = diag(I_2, B_1, I_2, \cdots, I_2)$, $A_4 = diag(I_2, B_2, I_2, \cdots, I_2)$ and so on up to $A_{n-1} = diag(I_2, \cdots, I_2, B_1)$, $A_n = diag(I_2, \cdots, I_2, B_2)$. It is now easy to check that $(A_1, A_2, \ldots A_n)$ is hypercyclic and furthermore that the set $HC((A_1, A_2, \ldots, A_n))$ is

$$\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_{2j-1}^2 + x_{2j}^2 \neq 0, \forall j = 1, 2, \ldots, k\}.$$

Case II: $n = 2k + 1$ for some positive integer k. If $k = 1$ the result follows by Proposition 3.3. Suppose $k > 1$. For simplicity we treat the case $k = 2$, since the general case follows by similar arguments. By Proposition 3.3 there exist C_1, C_2, C_3, 3×3 matrices such that (C_1, C_2, C_3) is hypercyclic. Let I_3 be the 3×3 identity matrix. Define $A_1 = diag(B_1, I_3)$, $A_2 = diag(B_2, I_3)$, $A_3 = diag(I_2, C_1)$, $A_4 = diag(I_2, C_2)$ and $A_5 = diag(I_2, C_3)$. It can easily be shown that $(A_1, A_2, \ldots, A_5)$ is hypercyclic. The details are left to the reader.

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